THE MATHEMATICS TEACHER

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THE MATHEMATICS TEACHER

Devoted to the interests of mathematics in Elementary and Secondary Schools

This organization has for its object the advancement of mathematics teaching in junior and senior high schools. All persons interested in mathematics and mathematics teaching are eligible to membership. All members receive the official journal of the National Council—THE MATHEMATICS TEACHER—which appears monthly, except June, July, August and September.

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J. KEPLER.
né en 1571. mort en 1631

THE MATHEMATICS TEACHER

Volume XXIV Number 3

Edited by William David Reeve

High School Finishing Mathematics Second Year

By Horace C. Wright
Deerfield-Shields High School
Highland Park, Illinois

Aim: To become acquainted with plane geometry, intuitively, concretely, and inductively by means of observation, scale drawings, and mensuration.

This course is to be a high school finishing course for second year Z students. It will contain the minimum assumptions, constructions, and theorems possible for foundational demonstrative geometry.

Unit of Work—Activity Objectives

I. To become aware of geometrical facts inferred intuitively and to exercise the spatial imagination.
Time, 2 weeks.

II. To measure approximately and to use the measurements for computation.
Time, 2 weeks.

III. To practice precise and succinct statements, which will continue throughout the course.
Time, 2 weeks.

IV. To draw and recognize:

A. Congruent triangles, perpendicular bisectors, and angle bisectors.

B. Arcs, angles, and chords in circles.

C. Parallel lines, and relative angles and parallelograms.

D. Angle sum in triangle and in polygon.

E. Secants and tangents to circles with relative angles; regular polygons.

F. Similar triangles and similar figures.

Time, 3 weeks.

V. To appreciate the meaning of:

A. Loci

B. Symmetry

C. Function idea.

Time, 1 week.

VI. Assumptions and theorems for informal treatment:

*1. Through two distinct points it is possible to draw one straight line, and only one.

2. A line segment may be produced to any desired length.

3. The shortest path between two points is the line segment joining them.

4. One and only one perpendicular can be drawn through a given point to a given straight line.

5. The shortest distance from a point to a line is the perpendicular distance from the point to the line.

6. From a given center and with a given radius one and only one circle can be described in a plane.

7. A straight line intersects a circle in at most two points.

8. Any figure may be moved from one place to another without changing its shape or size.

9. All right angles are equal.

10. If the sum of two adjacent angles equals a straight angle, their exterior sides form a straight line.

11. Equal angles have equal complements and equal supplements.

12. Vertical angles are equal.

13. Two lines perpendicular to the same line are parallel.

14. Through a given point not on a given straight line, one

straight line, and only one, can be drawn parallel to the given line.

15. Two lines parallel to the same line are parallel to each other.
16. The area of a rectangle is equal to its base times its altitude.

Time, 2 weeks.

VII. Construction

*1. Bisect a line segment and draw the perpendicular bisector.
2. Bisect an angle.
3. Construct a perpendicular to a given line through a given point.
4. Construct an angle equal to a given angle.
5. Through a given point draw a straight line parallel to a given straight line.
6. Construct a triangle, given (a) the three sides; (b) two sides and the included angle; (c) two angles and the included side.
7. Divide a line segment into parts proportional to given segments.
8. Given an arc of a circle, find its center.
9. Circumscribe a circle about a triangle.
10. Inscribe a circle in a triangle.
11. Construct a tangent to a circle through a given point.
12. Construct the fourth proportional to three given line segments.
13. Construct the mean proportional between two given line segments.
14. Construct a triangle (polygon) similar to a given triangle (polygon).
15. Construct a triangle equal to a given polygon.
16. Inscribe a square in a circle.
17. Inscribe a regular hexagon in a circle.

Time, 2 weeks.

VIII. Theorems

*1. Two triangles are congruent if (a) two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other; (b) two angles and

* **All from National Committee on Reorganization of Mathematics**

a side of one are equal, respectively, to two angles and the corresponding side of the other; (c) the three sides of one are equal respectively, to the three sides of the other.

2. Two right triangles are congruent if the hypotenuse and one other side of one are equal, respectively, to the hypotenuse and another side of the other.
3. If two sides of a triangle are equal, the angles opposite these sides are equal; and conversely.
4. The locus of a point (in a plane) equidistant from two given points is the perpendicular bisector of the line joining them.
5. The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines.
6. When a transversal cuts two parallel lines, the alternate interior angles are equal; and conversely.
7. The sum of the angles of a triangle is two right angles.
8. A parallelogram is divided into congruent triangles by either diagonal.
9. Any (convex) quadrilateral is a parallelogram (a) if the opposite sides are equal; (b) if two sides are equal and parallel.
10. If a series of parallel lines cut off equal segments on one transversal they cut off equal segments on any transversal.
11. (a) The area of a parallelogram is equal to the base times the altitude.
    (b) The area of a triangle is equal to one half the base times the altitude.
    (c) The area of a trapezoid is equal to half the sum of its base times its altitude.
    (d) The area of a regular polygon is equal to half the product of its apothem and perimeter.
12. (a) If a straight line is drawn through two sides of a triangle parallel to the third side it divides these sides proportionally.
    (b) If a line divides two sides of a triangle proportionally it is parallel to the third side. (Proofs for commensurable cases only.)
    (c) The segments cut off on two transversals by a series of parallels are proportional.

13. Two triangles are similar if (a) they have two angles of one equal, respectively, to two angles of the other; (b) they have an angle of one equal to an angle of the other and the including sides are proportional; (c) their sides are respectively proportional.
14. If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.
15. The perimeters of two similar polygons have the same ratio as any two corresponding sides.
16. Polygons are similar, if they can be decomposed into triangles which are similar and similarly placed; and conversely.
17. The bisector of an (interior or exterior) angle of a triangle divides the opposite side (produced if necessary) into segments proportional to the adjacent sides.
18. The areas of two similar triangles (or polygons) are to each other as the squares of any two corresponding sides.
19. In any right triangle the perpendicular from the vertex of the right angle on the hypotenuse divides the triangle into two triangles each similar to the given triangle.
20. In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.
21. In the same circle or in equal circles, if two arcs are equal, their central angles are equal; and conversely.
22. In any circle angles at the center are proportional to their intercepted arcs. (Proof for commensurable case only.)
23. In the same circle or in equal circles, if two chords are equal their corresponding arcs are equal; and conversely.
24. (a) A diameter perpendicular to a chord bisects the chord and the arcs of the chord. (b) A diameter which bisects a chord (that is not a diameter) is perpendicular to it.
25. The tangent to a circle at a given point is perpendicular to the radius at that point; and conversely.
26. In the same circle or in equal circles, equal chords are equally distant from the center; and conversely.
27. An angle inscribed in a circle is equal to half the central angle having the same arc.
28. Angles inscribed in the same segment are equal.
29. If a circle is divided into equal arcs, the chords of these

arcs form a regular inscribed polygon and tangents at the points of division form a regular circumscribed polygon.

30. The circumference of a circle is equal to $2\pi r$. (Informal proof only.)
31. The area of a circle is equal to πr^2. (Informal proof only.)

Time, 20 weeks.

Home Work

I. A. Collecting materials for scrap book.
   B. Listing of shapes in nature, industry, and household articles.
   C. Reading collateral material, i.e., uses of geometry, biographical matter, historical anecdotes.
   D. Preparing descriptions of geometrical uses found, and their illustration.
   E. Formulating brief floor talks.
   F. Making models.

Pupil Equipment

1. Note book covers.
2. Unlined and centimeter paper.
3. Ruler divided into inches and centimeters.
4. Compass.
5. Protractor.
6. No. 3 pencil and eraser.
7. Drawing board.
8. T Square.
9. 45°, 45°, 90° and 30°, 60°, 90° triangles.
10. Copy of a text to be chosen and to be used for reference and collateral reading.

Room Equipment

1. Section of squared blackboard slate.
2. 6 blackboard compasses.
   6 blackboard protractors.
   30 straight edges.
3. Locker cupboard for equipment of room and of pupils.
4. Collateral reading matter: books and magazines.

MYSTICAL MATHEMATICS
The Message of the Spiral

By Sister Alice Irene
College of St. Catherine, St. Paul, Minnesota

In 1798 when M. Boussard, a restless officer of Napoleon's army in Egypt, seeking relief from the boredom of inactivity, accidentally stumbled upon the Rosetta Stone, he unconsciously turned the page of a new epoch in human history. The secrets of a brilliant but long dead civilization were about to be made known. Soon was the world to know of the comings and goings of peoples and nations; of the doings and sayings of masters and of serfs; of the loves and the hates and the ambitions of courtiers, of kings and of queens.

How unutterably long must have seemed the years to that young officer as he waited for the interpretation of the pictures on that slab of stone! At last, after more than twenty years of patient and strenuous labor, Champollion found the key—the door was unlocked—it was swung open wide, and man read the stories of that mysterious past. Man still continues to read, for the records of three thousand years are not read in a day. He reads—ponders; he reads again, and he learns from the history of that brilliant age the lessons of life—Whence, Whither, Why.

And then came another memorable day—the day on which Sir Henry Rawlinson revealed to an eager age the long veiled lustre of twenty-five hundred years of Babylonian glory. About the same time that Columbus made his westward voyage, a Venetian by the name of Borbero returned from an eastern tour and told of a curious language which he had found carved on the walls of the temple of Shiraz in western Asia. The same curious wedge-like figures were also found, he said, on countless pieces of baked clay which lay buried in heaps of sand and débris. He talked, and the eager townsfolk listened, and the simple folk rejoiced, for was not this close to the land of mystery and wonder, Mesopotamia, the "country between two rivers," the Paradise of the Old Testament? Now, soon would they know of the secrets of Eden! In the very language of Eden, they would be told the things for which they lived. They would learn

now, beyond gainsaying, the "Whence, the Whither, and the Why". But years went by and man forgot—other sailors roamed the sea, other travelers found still other tablets of clay: old desires, and longings and expectations were revived only to be passed over and forgotten until, in the year 1850, Sir Rawlinson said to the world, "Behold the key to the Babylonian Age." He had given to an expectant people not only the complete alphabet of the Old Persian cuneiform containing 39 phonetic sounds but he had also given them the Babylonian alphabet containing upwards of 500 signs. Now in truth, could the city-mounds of the "country between two rivers" begin to speak and tell the story of 2500 years of life among the peoples who had successively dwelt in the one-time region of the Garden of Eden. But did the story they told help men more efficiently to find an answer to the eternal Whence, Whither, and Why?

There is still another epoch-making day in the history of this grand, good world of ours; epoch-making again because of the discovery of a famous inscription—an inscription which resembles, yet in many respects, differs from the two already described. Like the others, it, too, is symbolic, though it consists of only one symbol. It is inscribed not on stone nor on clay, but it is written in nature, not by the hand of a creature, but by the omnipotent Hand of the Creator. Mathematicians have named it the spiral.

About 200 years previous to our era, Conon of Samos sojourned in Egypt. He had been called there to teach in the great University of Alexandria. One morning as was his wont he walked on the bank of the river. Times without number he had watched the natives at work plaiting baskets of rushes which grew near the stream. But today an irresistible impulse urged him to draw near to a group of happy, youthful toilers. How deftly they worked! A few choice reeds were held together at their centers, tied, and the ends then spread out like spokes in a wheel; a long, flexible reed was then fastened to this center and the work of the weaver was about to begin. Nimble fingers plied it in and out among the spokes, and around and around again, the coil grew larger and larger—would it never end? It began at a mere point in the center; could it be made to go on and on, to go on and on for ever? Conon looked at the coil, but rushes and workers had vanished. He was alone with his thoughts. He was face to face with the Idea. As with Plato, he had ceased to see the concrete image in nature; he saw *the abstract idea,* the *geometric*

figure of the spiral. It seemed to express the end of his search. "I shall write to my pupil Archimedes; he shall prove the truth of my theorem; he shall interpret for Greece the meaning of this symbol."

While we wait for Archimedes to demonstrate the truth of Conon's theorem, while he with Euclid looks on "beauty bare", let us look, and looking try to see in the patterns of His creatures the *idea* of the Creator. To see these patterns we shall open the great book of nature and as we turn the pages of time, we shall look not upon the hieroglyphics of the Egyptians nor yet the wedge-shaped figures of the Assyrians, but upon the "hand-writing of the Finger of God."

Before opening this great book of nature we must turn aside a while and hear the story which the watchers of the sky have to tell. They tell us that if we point our telescope to the region of the milky-way, marvels of beauty will meet our gaze. They tell us that in this section of the sky more than 100,000 such luminous bodies can be seen. These bodies are called nebulae and perhaps most of them are spiral in structure. Those best known, and of the most perfect spiral formation are Messier's 51, the spiral in Draco, Andromeda, and the spiral in Ursa Major. They tell us again that a nebula is a universe in the making. In that instance, our own solar system must have at one time presented the same wondrous spectacle, and at a still farther distant time the great nebula which gave birth to the universe must have glowed with a splendor far out-shining any creation which our weak imagination could devise. The time of the appearance of this stupendous figure of glory is that period which we refer to as the "Beginning". When, after matter had been called into existence by the all powerful word of the Creator, God breathed on it, "order was brought out of chaos" and matter began to revolve in the form of the *spiral.*

The book of nature is now opened, and we have looked upon the first symbol in the "pageantry of the sky." We must recognize in the *spiral* the first written symbol in God's communication with man. He has traced it in letters of gold in the blue vault of heaven. It is seen through a veil of filmy clouds.

But only to a few is it given to look upon this vision of gold. To most of us the words of the poet or the brush of an artist must form the medium through which we gaze upon its beauty. However, as if in compensation for this loss, the symbol of the spiral was brought down to earth, and as we turn the first page of time, we shall see the

spiral as the Creator Himself has fashioned it for us in the minerals of the earth.

We are told that in the Beginning, matter moved in the form of a spiral; gradually the heavenly bodies were formed and they moved in elliptical paths about a larger mass. As they cooled, the minerals of which they are composed acquired certain habits; their molecules combined and arranged themselves in various figures; significant among these we may again see the spiral. The crystal formed by the molecules of sulphur is an exquisite, though microscopic, spiral described in a plane, while that of prochlorite is a conical helix well extended into the third dimension.

From the mineral we pass to the vegetable world, and here we are surrounded by spirals of every description. No longer need we depend on the words of the favored nor the lens of the microscope, but we may see with our eyes, touch with our fingers, and hold with our hands. The spirals on the third page of time are singularly our own. The Divine Scribe seems to grow more lavish; to right and to left, high up and low down, in secret and in the open, He writes His favorite symbol. He is writing a message for all and he desires that all should read. From the beginning of time, the leaves of numberless plants have arranged themselves in spirals around the stems of the plant, and if we look closely we can see the spirals in the arrangement of the petals of numberless varieties of flowers. The frond of a fern is a perfect spiral. Do you recall those happy childhood days when you spent hours in the woods in search of shepherd's crooks or fiddleheads? How sweet to think that you loved the spiral then! Was it an unconscious following in the "trails of glory" brought from your erstwhile home? Soon came the summer and as you gathered huge blossoms of "Queen Anne's Lace" a little observation would have revealed for you a spiral made up of spirals of wax-white lace. Then came, each in its own good time, the flowers of the garden. In the corolla of the fuchsia we find a spiral arrangement, each petal being rolled over an adjacent one and under the other; the "Rose of Sharon" shows another spiral convulute as does also the warm wine-red edge of the new canna shoot. The flower spike of "Love-Lies-Bleeding" is a graceful inverted conical spiral, while "Lady's Tresses" are examples of the up-right helix. The tendril of the sweet-pea is a cylindrical spiral as is also that of the hop, the grape, and the gourd, with all of which you have long been familiar. But have you ever watched

the movement of a bright scarlet runner? One can almost see it grow as it reaches out and twines itself around its nearest support.

In the fall comes the season for colored leaves and fruits. The sunflower has discarded her petals of gold, but her heart is bursting with fruit. On examining it closely you will find that radiating from the center in truly logarithmic fashion are numbers of spirals made up of closely set cells, each filled with its burden of a well-ripened seed. Almost the same can be said of the seed-vessels of a cone. Scientists have shown that the cone of each species contains a series of intersecting spirals, characterized by the number of seeds in each respective pair of spirals. Thousands of other examples of spirals exist, but I shall touch on only one more—that which obtains in the edges of leaves.

The elm and the begonia furnish well-defined specimens of spirals in the edges of their leaves. The terminal leaf of a branch of an elm gives an example of two symmetric spirals having their vertices at the base of the leaf and intersecting at the tip; in most of the other leaves are found examples of asymmetrical spirals similarly arranged, but varying in their proportions. Other splendid examples of asymmetric arrangement of spirals are seen in certain species of begonia. The leaf of the wax begonia is much like that of the elm in its contour, that of the snail exhibits the same arrangement, but each spiral makes more than one turn at its vertex. In the leaf of the Rex Begonia two asymmetric spirals are found; the vertex of the smaller being covered by the first complete turn of the other.

We have now seen the spiral as it appears on land, in flowers, plants, trees, and shrubs. We shall now look at the spiral as it appears in the depth of the sea. As Da Vinci in his *Note Book* so beautifully says, the Divine Architect who fashioned the land and sea knew well the power of the wind and wave. He knew that His children on earth would one day read the message He wrote in the waves. To read this message we shall look upon some of the most perfect spirals in nature—the shells of the sea. These may be found as fossils in the plain of the lowland or on the mountain side, or as fresh gleaming shells washed out by the tides. In each instance numberless examples abound.

Among the helical spirals in the fossil world are: the Red Whelk found in the cliffs of Felixtowe; the Volute Vespertilio, abundant along the coast of Hampshire; and Turrilites found in the sandstone of the

Island of Portland. The ammonite found in the cliffs of Lyme Regis is a perfect illustration of the spiral in a plane. Though thousands of years of sand and stone have robbed these shells of their lustrous splendor, their beauty of form is left unchanged. With mathematical exactness their spirals may be traced. The Ammonite reveals a plane coil of two and one-half turns; the whelk is a conical spiral of seven or eight turns with a height about equal to its base; and the Turrilites furnish perfect examples of the tall, slender, and many-coiled spiral whose height commonly measures from three to five times its base.

When we look at the living shell as it lies on the beach, we are in the presence of some of the most enchanting things in the realm of beauty. In the making of this symbol, beauty of form is combined with color and light. Each shell is a symbol, each symbol a structure of alabaster or of pearl which gleams and glistens with opalescent light.

In this realm the Murex Shells, Lop Shells, Volutes, and Mintre Shells are wonderful examples of helical spirals with height about equal to the base. In Screw Shells, Spindle Shells, Wentletraps, and Ladder Shells the spiral is a slender, tapering, and turreted dome. The Chambered Nautilus or "Ship of Pearl" is perhaps our most beautiful illustration of a plane spiral of many turns, while its "companion of the sea," the Paper Nautilus or Argonaut, so called by the Greeks and Romans because of an allusion to the fabled Argonauts who sailed away to find the Golden Fleece, is our fairest example of the one turned logarithmic spiral.

There is still another section on this third page of time. A little observation will find the spiral clearly defined in the birds of the air. The body outline of a bird at rest is a pair of asymmetric spirals, the larger one outlining the throat and chest, the smaller one having its apex at the base of the crown. But when, as is its wont, the bird suddenly turns its head through half of a complete revolution, its body remaining in the original position, the outline of the spirals is changed: The larger spiral then begins with the tip of its beak and includes in its sweep, crown, throat, and chest. Again when our friend is at work in the important act of capturing his traditional breakfast, the order of spirals is reversed. The larger spiral is then above and it includes the entire length of the bird, from the tip of his beak even to the end of his tail.

The spiral is also seen in the wings of birds. The outline of each

wing is a spiral in a plane; when the bird is in motion, the outstretched wings become a pair of helical spirals whose functioning power perhaps furnished the inspiration for none other than the Archimedian principle of motion. What could rival in beauty the spiraled motion of the white-winged sea gull as, rising from the swirling waters, it soars aloft into the clear blue sky and then as fitfully drops down to skim the waves? Or what more majestic than the spiraling motion of the outstretched wings of our country's emblem, the eagle, as it mounts higher and higher until from the crest of its mountain home, bathed in the first pink streaks of sunlight, it salutes the dawn of a new-born day?

The most attractive feature of many birds is found in their tail plumage. Here again I think we may say that the central charm lies in the spiraled structure of each feather. Perhaps the best known birds of this class are the lyre-bird of Australia, the Golden and the Silver Pheasant of China, the Black Cock of England and the several species of Birds of Paradise. The King Bird of Paradise is a flash of scarlet with two long tail feathers whose webbed ends are coiled into disks of emerald green; while the Great Bird of Paradise is a shower of sunlit spirals, falling from its shoulders and enveloping its entire body in a cloud of gold.

In the crested birds we have an additional spiral. The Blue Jay, The Cardinal Bird, the Tom-Tit and the Ivory-billed Wood-pecker are splendid examples of crests whose outline is that of a spiral, while the Lapwing, the Curassow, and the Cockatoo are the proud possessors of coronets each feather of which is a spiral of one or more turns.

And now as we turn to the fourth page of time, a cursory glance will discover to us the spiral in the animal world. Were I to touch on all the instances of spirals in this kingdom, I should be attempting an impossible task. I shall refer, therefore, to only a few of the most obvious examples with which all are familiar.

The general contour of the bodies of many animals abound in spiral curves. Recall for a moment the everchanging curves seen in the body of a squirrel as he frolics on the green; as he darts from limb to limb and having reached the topmost bough triumphantly balances himself, and raising his spiraled tail high above his head contentedly nibbles at the prize which he holds in his dainty paws.

Consider again our even more familiar friend, the dog. Not to mention the spiral curves of his head, back, and legs, have you ever

seen the greyhounds' expressive tail curled into a spiral disk? Or have you ever seen the Siberian Dog, posed as it were, with his tail saucily curled into a one-and-a-quarter turned spiral?

A little study of Leonardo da Vinci's "Cavallo" and Albert Durer's horse in "the Knight, Death, and the Devil" will convince you that both these artists felt the significance of the spiral curves in the majestic mien of these patient steeds. The same may be said of Rosa Bonheur as you may clearly see in her studies of horses in "the Horse Fair" and in "the American Mustangs." And an example of the same inspiration is seen in Anna Vaughn Hyatt's fiery steed in the Joan of Arc statue, on Riverside Drive in New York.

Countless other examples of these curves, as you no doubt are beginning to see, may be found in the animal world, but I shall cite only one more, namely, the spiral as seen in the horns of animals. The horns of animals occur in pairs. They are invariably symmetrical. The helical spiral is perhaps most frequently found, but there are many instances of the flat and the cylindrical spirals as well.

The horns of the wild goat of India consist of two straight screw spirals, measuring thirty-six inches in length, and a little less from tip to tip; the horns of the Merino Ram and the Tibetan Argali are helical curves, those of the Highland Ram are Cylindrical, while the wild sheep of Sardinia carry a pair of spirals in a plane. The horns of certain species of Albanian sheep are unique. In them we find a combination of plane and helical curves. The horn begins with a flat spiral of one complete turn and ends with a helical twist.

On this fourth page of time, there is another realm in which the Divine Scribe engraved His symbol. This realm is the kingdom of man. Here we find the spiral in all its beauty. Is it not fitting that the King of Creation in his "form Divine" should carry this mystical symbol in all its perfection?

Though scientists and anatomists tell us with ever increasing interest of the spiral structure of bones and sinews and internal organs of the human body, I shall call your attention to those curves only which are visible to all. Now, to describe the curves of the human body is a futile task, and since "beauty" exists only in the mind which contemplates it I shall try to bring to your minds a few of those studies with which you are familiar and which best illustrate the ideas which I should wish my words to convey.

The first of these productions is none other than the Venus de Milo.

Just a cursory glance at this famous piece of sculpture will make clear to you the meaning of the phrase "spiral curves in the human form." Henry Oliver Walker's representation of "Yesterday, Today, and Tomorrow," found over the west stairway in our own State Capitol, is another admirable illustration of the same idea. In the Congressional Library Mr. Walker has still another splendid delineation of spiral curves in his interpretation of Ganymede being borne to Olympus by Jove in the Form of an Eagle. In the study of Melpomene by Simmons, the artist has winsomely depicted in his cherubs the spiral curves as seen in the human form at infancy.

Another set of spirals worthy of careful attention are those found in the arms and hands. An illustration of these curves may be seen in F. W. Benson's study of "Autumn," one of a group of pictures representing the seasons, found again in the Entrance Pavilion of the South Corridor of the Congressional Library. Autumn is personified by a woman in flying draperies, in the midst of falling leaves; her left hand with the arm visible to the elbow resting on her bosom is a charming portrayal of the exquisite curves in the hand and arm. Having studied these curves as represented in this picture, try your own skill as an artist at producing, in shadows on the wall, similar curves made by flexing the joints of your fingers and wrists.

To realize the variety and beauty of the spiral in head, face, and neck, an initial study of the head and shoulders of Apollo or of the Venus de Milo might be made with singular profit. You will find the most delicate curves abounding in lips, eyes, nose, ear, and cheek.

Then study the faces of your friends—you will find them wreathed in spirals of lovliness—of ever-changing beauty—spirals breathing serenity in moments of calm—spirals which exult in moments of joy and spirals which weep in moments of grief.

But—have you ever noticed the spirals of gold on a baby's head? The wayward ringlets against a maiden's cheek? The strong black curl on a youthful brow? Or the silvery lock on an old man's head?

Thus in "Nature, that universal and public Manuscript that lies expanded unto the Eyes of all" we have read the first symbol in the "handwriting of God." We have found the spiral in the first cosmic figures of the heavens; in minerals; and in plants and shells and birds. We have found it in animals, and in its most beautiful conception we have found it in Man. And as did the readers of the hieroglyphics of Egypt and the cuneiforms of Assyria, we too ask

for its meaning. Does it contain in itself any real significance? Does it perhaps help *us* to answer that triple question of old, "Whence, Whither, Why?

Before answering this question let us consider briefly the reaction of previous ages of Man to the manifold occurrences of this symbol. That it caused them to look upon the spiral as of peculiar value, if not of supernatural significance, is evident from the findings of archeologists. Dachelette tells us that the children of the Magdelenians more than fifteen thousand years ago wore strings of helically spiraled shells about their necks as lucky charms. With spiraled shells they ornamented the forehead, arms, knees, and ankles of their dead. They laid their dead on beds of shells.

In a round barrow on Dunstable Downs, Mr. Worthington G. Smith excavated two hundred fossil echinoderms which surrounded the skeleton of a mother and her child. The fossils were known by the folk-names of "Heart-urchin" and "Fairy-loaf" or "Shepherds helmet." The Laplanders give us another instance of the use of sea-urchins in burial. They looked at snail shells as "dog-souls" and in their burials substituted them for the more valuable living quadrupeds.

The inhabitants of Travancore consider the Turbinella, a shell of helical form, as manifestation of the god Vishnu. It is used in the government seal as the national crest. The underlying idea in its use is that it brings prosperity and protection to its devotees. At weddings the moment of the ceremony is announced by a note sounded on a Turbinella, and an old habit of the country folk is to give their babies milk from the perforated end of the shell. The Turbinella is kept in the temple as a sacred object.

One more incident with which I wish to illustrate the mystical or supernatural meaning attached to this curve, relates to the cylindrical spiral. Among the Lithuanians, it is a time-honored custom that at the baptism of an infant the parents bury one of its ringlets at the bottom of a hop-pole, so that the child may "twine out of danger" in its life time, just as the spiral of the plant twines upward to the sun.

The reaction to the oft-repeated symbol is likewise evident in the art of the respective Ages of Culture. In the Paleolithic age, this reaction is represented in a fragment of Aurignacian art of more than thirty thousand years ago. It consists of a single and recurrent conventionalized spiral, deeply and clearly carved on the horn of a reindeer. Critics maintain that the spiral is depicted with a perfection comparable only to that of the Periclean Acropolis.

The reaction of the Neolithic age is represented by many and varied illustrations of the spiral engraved, with more or less perfection according to the period of the age, on stone. A splendid example of Neolithic art is seen in a large stone found in the New Grange tumulus, and another illustration of a later age is shown in a boundary stone found on the banks of the Boyne. In both these specimens the spiral appears in double conjunction while on the first mentioned it is found in triple conjunction as well.

The Bronze Age is rich in illustrations of spiraldic ornamentation as found in implements of warfare and in ornaments of metal used for personal adornment. A sword hilt of the earliest Bronze Age found in Scandinavia is encircled by two sets or joined spirals; in a heavy gold bracelet worn by a warrior a set of symmetric spirals is found at the termination at each end of the band.

In the middle Bronze Age, more delicate ornaments are found. Among these are cylindrical bracelets of as many as five double spiral twists, also worn by men, and similarly spiraled bracelets of greater delicacy for women, are found, in which the coil extends practically from the wrist to the elbow.

When we mention the Ionic Capital we have said all that is necessary regarding the reaction of the idea of the spiral volute in the culture of the Iron Age.

In concluding this reference to the spiral influence as manifested in art I shall recall to your mind a symbol which has been almost universally accepted as an emblem of good luck or good fellowship, or at least as a recognized motive in art; it is the Swastika.

In the *Curves of Life* by Theodore Cook, the author says, "In no instance do we find an abstract pattern that does not go back to nature by a series of conventional variations" and again, "the earliest pictures were not done for their own sake as 'works of art.' They were meant to convey an idea; they were supposed to retain some of the powers and qualities of the original."

Now conventionalized spirals as found throughout the ages are sometimes classified as being of Egyptian or Oriental influence. Hence the theory of their origin and meaning will enlighten us as to the reaction in question.

The Egyptian spiral traces its origin to the lotus. To the Egyptian the lotus signified creative power, strength, divinity, and every sacred phenomenon of life; that of the Oriental spiral, or the Swastika may be traced back to the logarithmic curves of the nautilus. The

idea it expresses is due to Chinese philosophy of distant historic times; it typifies the life-giving sun and is, therefore, the symbol of all that is good.

These considerations of the uses, the origin, and the meanings of the spiral of old take us back to our own question of Whence, Whither, and Why? Take us back to that day on the Nile when Conon sent his messages to Archimedes of Syracuse. When Archimedes' answer reached Egypt, Conon was dead. But in his treatise Archimedes defined the spiral as follows. "If a straight line one extremity of which remain fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line is revolving, a point moves at a uniform rate along the straight line, starting from the fixed extremity, the point will describe the spiral in the plane."

Now in the ordinary mathematical language of today, the idea of Archimedes' spiral is expressed by the equation, $\rho = k\theta$. This equation produces a curve which extends between zero and infinity, that is, it begins at zero, winds round and round its center from which it is ever receding, until it approaches infinity. It reaches up from zero to infinity—it reaches down from infinity to zero.

Now as someone has said of nature, may we not also say of mathematics, the scientific interpretation of nature, "The entire value of the beauty of this mathematical symbol (nature) is lost unless we conceive behind it One who has designed it—unless we are willing to sacrifice the aesthetic emotion in its highest development and in its greatest examples, we must believe in a Great Spirit, whose manifestations these things are."

In that instance may we not see in the spiral, that oft repeated manifestation of the Creator, a message of His love to man? A love which has come from Infinite heights through devious ways, to us, to nothingness, and which through equally devious ways has raised us up from nothingness, even to the Infinite?

Since Archimedes' interpretation of the symbol of the spiral, recognized as such on that greatest of epoch-making days, we need no longer say,

> "Drink, for you know not whence you came, nor why,
> Drink, for you know not why you go, nor where."

but we can with mathematical certitude, confidently say,—

> "I *know* Whence I came,—I *know* Whither I go,—and *Why*."

The Fusion of Plane and Solid Geometry*

By Joseph B. Orleans
George Washington High School, New York City

In discussing the fusion of plane and solid geometry one cannot help making reference to statements relating to this problem that have appeared in the past. For example, in 1901, at the meeting of the British Association for the Advancement of Science, when John Perry delivered his famous address on the teaching of mathematics, one of the members of the audience who took part in the discussion said:

> Geometry is perhaps the subject which suffers most from inefficient teaching. I do not agree that we should abolish Euclid from the schools. Euclid's methods are very valuable as specimens of sound reasoning and as illustrating the nature of a proof. The absurdity of our present system lies in using Euclid as a means of teaching geometrical facts. For that purpose the children should handle solid bodies and from them obtain plane figures. The faces of crystals supply many of the plane figures treated in Euclid, and obtaining the figures from natural objects gives life and interest to the subject such as cannot be induced by drawing lines arbitrarily on paper. When the names of the figures are familiar to the child and he has for himself observed and classified their properties, he may be interested in an account of Euclid and his times and be incited to find how the properties he has observed for the few objects he can handle can be shown to be generalized truths for abstract forms.

As one goes through sources of information on the history of geometry, one finds for example that in France D'Alembert (about 1784) in an article on the Elements of Geometry did not consider at all the division into plane and solid geometry, that Bertrand about the same time took the same attitude, that Méray in 1874 brought out a text in which there is complete fusion of the plane and the solid geometry, that in Germany about 1870 there appeared as part of a larger movement a number of texts in which the attempt was made to break down the division of geometry into plane and solid, that Felix Klein, as a leader in mathematical reform, among other things, encouraged the fusion of plane and solid geometry.

In describing the movement in France, A. W. Stamper in his *History*

* Read before the Mathematics Sections of the New York State Teachers Convention, Buffalo, N.Y., and of the New Jersey State Teachers Convention, Atlantic City, N.J., November 7 and 10, 1930.

of the Teaching of Elementary Geometry quotes the opinion of a professor who used Méray's book for several years in one of the normal schools:

> Previous to the use of the new method the beginning was slow, the demonstrations tiresome for the pupils, and often entangled with useless details; the spirit of the pupils was hampered by useless, at times pedantic quibbling. Under the pretense of making a complete proof, they were foolishly delayed, in order that they might follow a uselessly exact method. Now the lessons are animated and the pupils are interested in the new method. . . . . They advance surprisingly, their intelligence is lively, and they work very easily because of the new light which is presented to them.

The movement spread also through Italy, just as it did in France. The text of De Paolis appeared in 1884 and one by Lazzeri in 1887, which was followed in 1891 by the *Elements of Geometry* by Lazzeri and Bassani after they had tested the work in the classroom, and in 1897 by the text of Professor Veronese. In 1900, at the International Congress of Mathematicians in Paris, Veronese explained the simultaneous treatment of plane and solid geometry as follows:

> I think the beginning should be made with Rectimetry (geometry of the straight line), at least so far as relates to the principles. When the latter are well established, special theories, such as equivalence, similitude, measurement, etc. could be treated simultaneously in the plane and in space. I do not say with the fusionists that in these theories figures in three dimensions ought to intervene in the demonstration of planimetric theorems, but that numerous demonstrations can be extended simply from the plane to space. Such a mode of exposition will economize time and make the pupils grasp better the relations which exist between the diverse parts of the same theory.

The Report of the National Committee on Mathematical Requirements which appeared in 1923 sets down as the principal purposes of instruction in plane geometry—

> to exercise further the spatial imagination of the student, to make him familiar with the great basal propositions and their applications, to develop understanding and appreciation of a deductive proof and the ability to use this method of reasoning where it is applicable and to form habits of precise and succinct statement, of the logical organization of ideas and of logical memory.

Concerning solid geometry it says:

> The aim should be to exercise further the spatial imagination of the student and to give him both a knowledge of the fundamental spatial relationships and the power to work with them. It is felt that the work in plane geometry gives enough training in logical demonstration to warrant a shifting of emphasis

in the work on solid geometry away from this aspect of the subject and in the direction of developing greater facility in visualizing spatial relations and figures, in representing such figures on paper and in solving problems in mensuration. For many of the practical applications of mathematics it is of fundamental importance to have accurate space perceptions. Hence it would seem wise to have at least some of the work in solid geometry come as early as possible in the mathematics courses. Some schools will find it possible and desirable to introduce the more elementary notions of solid geometry in connection with related notions of plane geometry.

The report prepared for the British Mathematical Association on the Teaching of Geometry refers several times to the possibility of touching upon solid geometry together with the plane.

There is no reason why an individual proposition should not be known long before an attempt is made to deal systematically with its subject matter, and the rider that undertakes an incursion into territory which there is no immediate prospect of exploring thoroughly is most stimulating. In particular, details of solid geometry should become familiar far in advance of any systematic abstract study. For example, the theorem that "lines parallel to the same line are parallel to one another" can be asserted almost as soon of space as of a plane and leads to the conclusion that the midpoints of the sides of any quadrilateral determine a parallelogram, whether the quadrilateral is plane or skew.

In describing what the committee calls the Deductive stage of the geometry course, the report says:

The boy next learns to prove theorems and riders and to write out proofs. The subject matter of this stage is the whole elementary plane geometry with occasional inroads upon the easier parts of solid geometry.

Concerning solid geometry the report says:

In Euclid's sequence the very elements of solid geometry had to yield precedence to the study of elaborate plane figures with the unfortunate result that few boys brought up on the Elements ever learn more about three-dimensional space than can be gathered by observation and common-sense. In recent years, something has been done to remove this serious defect in the curriculum, but the committee is of the opinion that a good deal more is needed. The view held is, in brief, that there ought to be no formal separation between plane and solid geometry and that, although, for practical reasons, it is often necessary to deal with tri-dimensional properties, of figures apart from the corresponding plane properties, the two kinds should always be studied in close relation with one another. This maxim applies with particular force to the fundamental notions of congruence, symmetry, similarity and parallelism. Of all these it may be said that a beginner gains a much more full and vivid idea of their significance if they are at first illustrated by reference to solid rather than to plane figures. Common articles made to the same design,

such as books, chairs and houses give the best introductory illustrations of congruence, the two hands, as an example of symmetry, present an essential feature which is missed when the idea is based upon the comparison of plane figures capable of being superimposed; the relation of model to its original offers the best starting point for the discussion of similarity, and so on. A more substantial difficulty (in teaching solid geometry) is the perennial one of finding time for an addition to an already over-full program. One way to meet this is to apply the pruning knife to inessentials—a process which has not yet been carried sufficiently far. Another is to teach a good deal of new matter in the form of exercises as riders upon the ordinary propositions of plane geometry; the new material introduced in that way would add very materially to the interest and value of the teaching in plane geometry itself.

As a contrast I quote from David Eugene Smith's *Teaching of Geometry:*

There have been numerous suggestions with respect to solid geometry, to the effect that it should be more closely connected with plane geometry. The attempt has been made, notably in France and in Italy, to treat the corresponding propositions of plane and solid geometry together; as, for example, those relating to parallelograms and parallelopipeds and those relating to plane and spherical triangles. Whatever the merits of this plan, it is not feasible in America at present, partly because of the nature of the college entrance requirements. While it is true that to a boy or girl a solid is more concrete than a plane, it is not true that a geometric solid is more concrete than a geometric plane. Just as the world developed its solid geometry as a science long after it had developed its plane geometry, so the human mind grasps the ideas of plane figures earlier than those of the geometric solid.

This was written in 1911. Conditions in geometry classes are so far different after a lapse of twenty years. I feel that the British Report makes the recommendations that are suitable today.

The aims set up for solid geometry by the Report of the National Committee seem to be very much worth while for all high school pupils; yet very few of them make any contact with the solid geometry. In a large city high school like the one from which I come, with a register of 5,500 boys and girls of whom fully 4,500 take two or more years of mathematics, only 40 elect solid geometry. Corresponding figures may be quoted also for the other schools. The figures for the State of New York for 1928 show that while 109,198 pupils studied elementary algebra in 919 schools and 70,675 took plane geometry in 821 schools, only slightly more than four thousand were in solid geometry classes in 279 schools.

Since, therefore, solid geometry touches so few of the pupils and

since we accept the stand of the National Committee in regard to the value of solid geometry, we should certainly arrange to teach some of it to the pupils while we have them in our plane geometry classes. Is there time for this? As matters stand at present, teachers will say no. And justly so. We have all we can do to cover the ground in the plane geometry. The question, however, may be asked as to how much the pupils carry from their plane geometry as it is constituted at present. I am afraid that, *entre nous,* we must admit that they do not carry away from the year's work as much as we would like them to have acquired. There are many reasons for this, I think, (1) our false democracy in education which compels us to give the same course to all, with the attitude of "let him try" and "give him a chance" and then "give him another chance to fail a second time," (2) the method of high school organization which prevents homogeneous classification because of program exigencies, (3) the domination of extra mural examinations and (4) the poor teaching, not in all class-rooms, to be sure, but in a sufficiently large number to make the problem acute. Whatever the reason may be, however, the fact remains that the scholarship reports for various schools are not encouraging. The percentage of failure in geometry 1 in the New York City High Schools, for example, ranges from 21 to 36.

A better picture of the situation is obtained from the figures which tell how the final class marks are distributed. In New York City the passing mark is 65. Teachers of experience will agree that a mark of 70 indicates little more than 65. They may both be considered bare passing marks. Taking two schools from which it was possible to obtain the necessary figures:

	Geometry 1	*Geometry 2*
Per cent of pupils with a final mark of 65..........	30	33
Per cent of pupils with a final mark of 70..........	34	30
Per cent of pupils passing for the term.............	72	74

If a final mark of 65 or even 70 represents work that is barely passing, is it not significant that more than three-fifths of the group deserves no more than the bare passing mark; and this is true of geometry 2 as well as of geometry 1 after the failures have been eliminated?

I do not place so much emphasis on the results of the Regents examinations because the examinations are not standardized and, as a result, the figures for consecutive terms are not comparable, and

because I do not feel that the papers represent geometry as pictured in the aims listed by the National Committee. And yet let us see what the figures do say concerning the Regents Examinations in geometry in New York State in 1928.

Number of papers written	60,089	
Number of papers claimed	46,308	77 per cent
Number of papers accepted	43,245	72 per cent

and this after the poorer pupils are eliminated by failure at the end of the first half year.

This picture of the situation in geometry is not new. In 1901, after listing as "obvious forms of usefulness in the study of mathematics, (1) producing the higher emotions and giving mental pleasure, (2) brain development and producing logical ways of thinking, (3) the aid given by mathematical weapons in the study of physical science, (4) passing examinations, (5) giving men mental tools as easy to use as their legs and arms, (6) teaching a man the importance of thinking things out for himself, (7) making men in any profession of applied science feel that they know the principles on which it is founded, and (8) giving the acute philosophical minds a logical council of perfection," John Perry says, "At present, with the exception of (4) which is not particularly nice, in the performance of these functions mathematics affects less than one per cent of the boys who are supposed to study it." And later, "I believe that men who teach demonstrative geometry and orthodox mathematics generally are not only destroying what power to think already exists, but are producing a dislike, a hatred for all kinds of computation and, therefore, for all scientific study of nature and are doing incalculable harm." A rather strong statement, but one which contains a great deal of truth.

It should be possible to omit some of the plane geometry without weakening the course materially, in order to make room for enough of the solid geometry that is considered worth while. I have experimented with this in an unofficial way. I shall merely mention what I have found possible to do in my own classes.

First, however, let us understand the practical as distinct from the ideal situation. If a school system were definitely and truly organized on the 6-3-3 basis and the mathematics of the seventh, eighth, and ninth years included the intuitive geometry with possibly

a unit of demonstrative geometry, then it would be possible to devote one third of the tenth year to the solid geometry as outlined in the National Report. This is the ideal situation. But what is the practical situation? We have in New York City, for example, about fifty-three Junior High Schools and a State Junior High School Syllabus in mathematics. As far is I know, this syllabus is in force in no school in New York City. Whatever is being done in the seventh and eighth years, the ninth year course is the straight elementary algebra that is taught in the senior high school. Our pupils in the tenth year, therefore, come without any previous work in geometry that is worth anything.

If then some of the plane geometry is omitted, how much of the solid geometry can be introduced and in connection with which topics? The following notions can be developed together easily from the very beginning of the course:

A line is the path of a moving point.	A plane is the path of a moving line.
Through a point any number of lines can be passed.	Through a line any number of planes can be passed.
Two lines intersect in a point.	Two planes intersect in a line.

Just as we can think of a line parallel to a line and of a line perpendicular to a line, so we can also think of a plane parallel to a plane, a plane perpendicular to a plane, a line perpendicular to a plane and a line parallel to a plane. The pupils are surrounded by illustrations of these notions.

They become fixed by careful development and by constant reference to them during the early part of the geometry year. After the pupil has learned about congruent triangles, the proof of the theorem that "if a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to their plane" serves as excellent material for practice in demonstration. This calls for a number of facts from plane geometry.

Pupils learn quite easily that two angles on a plane whose sides are respectively parallel and run in the same direction are equal. They grasp just as easily that this is also true if the angles do not lie on the same plane. This proof offers drill with the facts about parallelograms and congruent triangles.

The best topic in connection with which one may step from the board into space is, of course, loci. This can be done in an elementary

form as soon as the pupils have learned the theorems about the perpendicular bisector of a line and about the bisector of an angle. Later on the complete discussion of loci and of the intersection of loci gives the children an opportunity to exercise their spatial imagination.

At the beginning of Book III, when they learn that "a line parallel to one side of a triangle divides the other two sides proportionally" and that "three or more parallels divide two transversals proportionally," they can also understand why "if two straight lines are cut by three parallel planes, the corresponding segments are proportional." When they have become acquainted with similar figures, they will be able to prove that "if a pyramid is cut by a plane parallel to the base, the lateral edges and the altitude are divided proportionately and the section is similar to the base." The proof of "the areas of two similar polygons are to each other as the squares of any two corresponding sides" leads naturally to the proof that "the area of a section of a pyramid parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid."

Book IV furnishes another point of contact at which the application of the area theorems of plane figures can be made to the lateral and total areas of the various solids, and in connection with which the volume theorems can be developed.

What I have mentioned does not include all the possibilities. Nor does it mean that the pupils are getting a course in solid geometry, but it is the best that one can do under the circumstances. Let the pupils get this small amount rather than nothing at all.

Where can time be saved from the plane geometry for the new work? The answer is to postulate more than we do. Do not have the pupils spend unnecessary time on book propositions the proofs of which depend upon auxiliary lines that they themselves cannot suggest. I do not mean to omit them altogether. After their proofs have been developed and demonstrated, their importance established and the fact learned, they should be applied in exercises but their proof need not be repeated and repeated just because they are numbered propositions. This will not weaken the logical backbone of the geometry nor interfere with the necessary sequence of theorems. That can be taken care of by questions like, "Upon what preceding theorem is the proof of this one based?" or, in reviewing the work of one or more weeks, "What theorems that follow does this one prepare for?"

The question of one year of plane and solid geometry is once more the problem of the day in the teaching of geometry. It has become a controversial issue. This is partly due to the fact that the opposing sides have not defined their terms. Even those who agree about the desirability of the combination disagree about the method of procedure to be followed in teaching the year's work, whether to have the solid follow the plane or to fuse the two. The decision should not be the result merely of discussion. It should be based upon experience in the classroom. That is the only way to evaluate any revision in a course of study. My little experiment leads me to prefer the fusion of the two.

Program of the Kansas Association of Mathematics Teachers

Topeka, Kansas, January 24, 1931

Morning Program

1. A Geometric Development of Laws Governing the Fundamental Operations with Positive and Negative Numbers.
   A. D. Dougherty, Kansas State Agricultural College
2. The Newer Type of Secondary Mathematics
   John A. Swenson, Chairman of the Department of Mathematics, Wadleigh High School, New York City.
3. The Unit in Mathematics...E. R. Breslich, University of Chicago

Afternoon Program

1. Developing Clear Meaning of Mathematical Concepts
   E. R. Breslich, University of Chicago
2. The Development of the Newer Type of Mathematics
   John A. Swenson, Wadleigh High School, New York City
3. Teaching a Combined Course in Plane and Solid Geometry in the Tenth Year.
   Minnie Stewart, Topeka High School

A Combined Course in Plane and Solid Geometry?

By Charles A. Stone
Laboratory Schools, University of Chicago

Probably no question is receiving more attention today than that of the reorganization of plane and solid geometry. This is due to the fact that many teachers are of the belief that solid geometry is declining in popularity as a secondary school subject, and hence are strongly urging that measures be taken to preserve it. Leaders in the teaching of mathematics maintain that if solid geometry is to be preserved, it can be done in three ways:

1. It can be taught as a separate unit in the tenth grade much as it is today, except that it must be more concentrated.

2. It can be fused with plane geometry at places where such fusions seem most desirable and possible.

3. It can be taught intuitively. Some teachers feel that the second method will solve the problem and advocate that a combined course in plane and solid geometry should be taught in the tenth grade. However, they do not present experimental evidence to show the desirability of adopting this plan, nor do they show that mathematics teachers as a whole are in favor of the plan.

The object of this paper is therefore two-fold:

1. To reflect the view point of the mathematics teachers of the Central Association of Science and Mathematics Teachers in regard to this problem.

2. It was decided to teach a combined course in plane and solid geometry to a tenth grade class in the University High School and through careful testing determine the advisability of teaching such a course at the tenth grade level.

The data for the first part of the experiment was obtained by mailing two hundred fifty questionnaires to the teachers of the above mentioned association. One hundred forty were answered and returned. The questions and tabulated results are given below. In replying to the questions some teachers justified their replies while others con-

tributed some very interesting comments. These are given below the tabulations for each question.

TABULATION AND RESULTS ON QUESTIONNAIRE

Question I

Do you feel that solid geometry should be taught in the High School?

Yes	108
No	6
As elective	22
For some pupils only	1
Not qualified to answer	3
Total	140

COMMENTS SELECTED FROM THE QUESTIONNAIRES

1. Every first class high school should have a course in solid geometry.

2. Solid geometry adds much to the thinking process.

3. The pupil is trained in some of the most valuable reasoning when he studies solid geometry.

4. The work with formulas and calculations found in solid geometry is badly needed.

5. Those who have solid geometry know plane geometry better.

6. Plane geometry is too restrictive. A knowledge of solid geometry is needed since we spend most of our lives in a three dimentional space.

Question II

Do you favor a one-year course in plane and solid geometry?

No	88
Yes	27
Undecided	4
As elective	4
Yes, if 1½ year course	4
Yes, if shown worth while experimentally	3
Yes, if school year is long enough	1
Yes, if not going to college	3
For some pupils	2
Yes, if preceded by proper course	1
Total	137

COMMENTS

The comments were largely reasons opposing a one year course in plane and solid geometry, and are given as follows:

1. Too many failures in a one year course.

2. The required course cannot be covered in the allotted time.

3. The organization is already too long and tiresome.

4. If plane geometry is well learned first, solid geometry can then be learned with less attention to form, reasoning and expression. Thus more attention can be given to mathematical truths and uses.

5. High school students are not mentally capable of doing a combined course in one year.

6. A combined course is too difficult for slow pupils.

7. A combined course is not practical as too much has to be omitted, or the work must be done slovenly.

Question III

If so, do you favor an organization that fuses or correlates the subject matter, or do you prefer to teach solid geometry after completing the plane geometry?

Fusion	27
Separately	18
Undecided	1
Total	46

Of the 27 who answered in the affirmative in question II without qualifying their answers, 20 were in favor of fusion. Question III did not excite a great deal of comment, but a few contributors stated that they could not see how it was possible to fuse plane and solid geometry.

Question IV

Do you feel that intuitive solid geometry as taught in the Elementary School or Junior High School is sufficient for pupils who do not plan to elect solid geometry in the Senior High School?

Yes	66
No	43
No reply	11
Yes, if not going to college	80
Yes, if not taking a technical course	2
Could not interpret question	3
For mensurational purposes only	2
Yes, if not taking further mathematics	2
Total	137

COMMENTS

The reasons favoring the teaching of intuitive solid geometry are given as follows:

1. The substitution in the formulae dealing with solids gives good practice in algebra.

2. Intuitive solid geometry is exceedingly helpful to slow pupils who will not take more mathematics.

3. The conception of space and space relationships can be taught here and will be valuable.

4. A course in intuitive geometry is a good foundation course.

5. It is better to teach the solid geometry in this manner than to crowd it into the tenth grade.

6. One person did fifteen majors in college mathematics successfully with but a Junior High School background of solid geometry.

Some teachers opposed the teaching of intuitive solid geometry and gave the following reasons for their opposition:

1. Intuitive solid geometry gives the pupil only a smattering of knowledge concerning the subject.

2. Teachers do not realize what they are doing in the teaching of intuitive solid geometry.

QUESTION V

If you do not feel that solid geometry should be taught in the High School, what mathematics would you substitute for it?

Plane trigonometry	9
Calculus, analytics, mathematical analysis	8
Arithmetic and one semester of algebra	2
Mechanical drawing	1
Arithmetic	1
Total	21

COMMENTS

1. There can be no better course both practically and culturally than solid geometry, which is most useful for both normal and super-normal boys and girls. Hence there should not be a substitution.

2. Solid geometry furnishes a good opportunity for enriching the student's knowledge of geometry.

3. Solid geometry should not be taught for a full semester. Part of the semester should be used to keep algebra alive.

By this time the reader certainly will be convinced that the majority of teachers in this study are in favor of keeping solid geometry in the high school curriculum. Thus there is no need to fear that it is declining in popularity as a high school subject and will be ultimately replaced insofar as this group herein represented is concerned. This has certainly been brought out by the replies given in question five. Here it is seen that plane trigonometry, calculus, analytics, and mathematical analyses are not at all favored as substitutes for the solid geometry. As far as the question of fusing plane and solid geometry is concerned the replies obtained in question number two show that teachers are strongly opposed to any such movement.

The writer is willing to admit that any deductions drawn from re-

plies to a questionnaire may be criticized on the score that the deductions are not always based on actual experience. However, he feels that the results obtained are worth while because it is well to know what the general attitude on the subject is. In order to offset criticism a different type of investigation was pursued. This constitutes the second part of the experiment.

Two tenth grade classes of almost equal ability, and a twelfth grade solid geometry class at the University High School were chosen for experimental purposes. Since the tenth grade classes were about to begin the study of the unit, "Parallel and Perpendicular lines," it was decided to use this unit in the experiment. The plane geometry taught in this unit is found in Breslich's *Senior Mathematics,* Book II, page 39.

The superior of the two classes was taught the solid geometry that could be fused with the plane geometry of the unit. Let us designate this class as class A. Thus, when the class studied the theorem, "If two parallel lines are cut by a transversal, the alternate interior angles are equal," the corresponding theorem in solid geometry was studied. The other class (class B) was taught the plane geometry only, while the twelfth year class (class C) was taught solid geometry only. The reason for this procedure is obvious. If the tenth year class studying both the plane and solid geometry could do the plane geometry as well as the class studying plane geometry only, and the solid geometry as well as the twelfth year class, then certainly it would be desirable to teach a combined course in plane and solid geometry.

Upon completion of the units, test II of the "Breslich Achievement Tests," was administered as the plane geometry test. As there is not in existence a solid geometry test for this particular unit only, the writer was compelled to construct one himself. The results of the tests are given as follows:

Class A (plane geometry)	.866
Class B (plane geometry)	.875
Class A (solid geometry)	.650
Class C (solid geometry)	.920

An examination of the above results shows that the superior of the two plane geometry classes lost ground in the plane geometry and made a very poor showing in the solid geometry. The loss in plane

geometry may be attributed to the reorganization of the unit to include the solid geometry, because in the past class A always made a better showing on unit tests.

While the writer feels that the teaching of a one year course in plane and solid geometry is an impossible program, he would like to propose that further experimentation be carried on before any conclusions are definitely reached. This could be carried on as follows:

1. Make a survey of the mathematics being taught in the high schools to determine just how solid geometry is faring.
2. Set up experimental centers in the various localities of the country and determine whether a combined course in plane and solid geometry is more efficient than the separate courses.

Let one small formula be quoted as an epitome of what Euler achieved $e^{\pi i}+1=0$. Was it not Felix Klein who remarked that all analysis was centered here? Every symbol has its history—the principal whole numbers 0 and 1; the chief mathematical relations $+$ and $=$; π the discovery of Hippocrates; i the sign for the 'impossible' square root of minus one; and e the base of Napierian logarithms.—H. W. Turnbull, *The Great Mathematicians,* Methuen & Co., Ltd. London, 1929, p. 100.

An Effective Method of Teaching Pupils How to Solve Verbal Problems

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High school pupils who study algebra have greater difficulty in the solution of verbal problems than with the study of any other topic in that subject. This situation may be explained partly by the fact that the solution of problems requires something more than a mere knowledge of skills developed in algebra. This something, as generally understood, is the ability to reason. It is this ability to reason which is a vague thing, and which teachers of high school algebra have failed to study carefully and analyze into its component parts.

Reasoning is not a mysterious process, but is the organization of facts and skills already at hand, to arrive at a desired end. That individual reasons best who has at his command a very large number of facts and skills, and who has used them extensively in a variety of situations. If we apply this definition of reasoning to the solution of verbal problems, it follows that a pupil can improve his reasoning ability, and, therefore, his ability to solve problems, provided he is equipped with the necessary skills and facts with which to reason, and has had training in organizing these facts and skills.

Teachers and text-books are much to blame for the difficulty that a pupil has in the solution of verbal problems. The customary procedure seems to be to confront the pupil with several pages of such problems, preceded by an inadequate list of illustrations. If two or three illustrations are given, they are followed by exercises falling into six or seven distinct types, some of which of necessity have not been illustrated at all. The pupil is then expected to solve all the exercises by means of the illustrations and any additional instruction given by the teacher. Little wonder that our pupils have difficulty, because the method of instruction and of textbook presentation is as a rule the least adequate and the least satisfactory of any of the topics of high school algebra.

The various problems which the student is called upon to solve in high school algebra fall into some half dozen distinct types, such as uniform motion problems, mixture problems, work problems, number

problems, profit and loss problems, geometry problems, and digit problems. Each group makes use of a body of special facts and abilities a knowledge of which is essential to the solution of problems of that type. Occasionally these facts and abilities apply to problems of another group, but more frequently there are at least some items which are peculiar to the solution of only one type of problem.

It seems, therefore, that a more desirable method of treatment would be to study problems by types, presenting carefully to our pupils the fundamental facts and relationships common to such types. This should be followed by a careful analysis of the problem, after which numerous exercises should be solved to fix the distinctive features of that type of problem. When all the various types have been thus considered, a miscellaneous list should be presented. A pupil should be encouraged to classify the problems in such a list by types before he begins the solution of any particular problem. When this has been done, the pupil should follow the plan for solving that particular type of problem. The remainder of this paper is devoted to illustrating at some length just what is meant by this procedure.

Type 1. Uniform Motion Problems. Let us assume that we are about to study uniform motion problems similar to those here listed.

a. Two airplanes start at the same time from Chicago for Denver. If one flies at a rate of 100 miles an hour and the other at a rate of 80 miles an hour, in how many hours will they be 100 miles apart?

b. Two men travel in opposite directions from the same point. The rate of one is two-thirds that of the other. After the man traveling at the slower rate has traveled 4 hours and the other man 6 hours, they are 28 miles apart. Find the rate of each.

c. A man can row downstream twelve miles in the same time that he can row four miles up stream. If the rate of the current is $2\frac{1}{2}$ miles an hour, find his rate of rowing in still water.

The development leading up to the solution of problems of this type should then be as follows:

A. Providing a Knowledge of Fundamental Facts. These facts are as follows:

1. *Ability to express in symbols the words and phrases of motion problems.* Although this ability may not be taught at this point for the first time, it should certainly be reviewed here. The pupil should be able to write suitable symbols for such expressions as distance, rate, time, two-thirds of the rate, the rate increased by n, and other similar expressions.

2. *Ability to use the uniform motion formula to express one variable in terms of the others.* The pupil should next be able to use the formula $d = rt$ to answer such questions as the following:

a. How far will an automobile travel if it runs:
5 hours at 25 miles an hour?
n hours at 40 miles an hour?
8 hours at x miles an hour?
n hours at x miles an hour?

b. How long will it take a man to travel:
120 miles at a rate of 30 miles an hour?
60 miles at a rate of x miles an hour?
250 miles at a rate of $x + 4$ miles an hour?
n miles at 25 miles an hour?
x miles at r miles an hour?

c. At what rate will a man travel if he goes:
60 miles in 3 hours?
240 miles in n hours?
n miles in 8 hours?
m miles in x hours?

d. If a man rows 4 miles an hour in still water, and the rate of flow of water in a stream is x miles an hour, what will be his rate of travel:
If he is rowing upstream?
If he is rowing downstream?

B. How to Analyze the Problem. A pupil must next be taught what the important items in each problem are and how they may be expressed. It is advisable to supply the pupils with several problems where the questions listed are as detailed as in the following illustrations.

Illustaration 1. Two automobiles travel toward each other from towns 350 miles apart, one traveling at an average speed of 30 miles an hour, the other at an average speed of 40 miles an hour. In how many hours will they meet?

a. How can we represent the number of hours it takes the automobiles to meet?

b. How far will each automobile travel in this time?

c. How can we represent the total distance traveled in terms of the answers to (a) and (b)?

d. Since the problem states that the total distance traveled is 350 miles, write an equation stating this fact.

Illustration 2. A man walked to a certain point and rode back. The total time of the trip was 3 hours. He returned at a rate of 20 miles an hour, and walked at a rate of 4 miles an hour. How many miles did he travel?

a. How can we represent the number of miles he traveled?

b. How can we represent in terms of the answer to (a) the distance he walked? rode?

c. What was the rate of walking? riding?

d. How, in terms of the answers to (b) and (c), can we represent the time spent in walking? in riding?

e. What, in terms of the answers in (d), is the total time spent on the trip?

f. Since the problem states that the total time is 3 hours, write an equation from the above facts.

Illustration 3. A man rows downstream a certain distance in 8 hours and returns in 12 hours. His rate downstream is 2 miles an hour more than his rate upstream. How far did he row downstream?

a. How can we represent the distance rowed downstream?

b. What was the time taken to row downstream? upstream?

c. Use the answers to (a) and (b) to find the rate of rowing downstream; upstream.

d. Write an expression showing the difference in the rates of rowing, using the answers in (c).

e. Since the problem states that the difference in the rate of rowing upstream and downstream is 2 miles an hour, write an equation from these facts.

C. Organizing the Facts to Solve the Problem. From the preceding exercises it is apparent that the fundamental formula underlying all uniform motion problems is $d = rt$. The facts given in such problems enable us to form an equation on one of the following bases:

1. Rate Equation—Two equal expressions for the rate.
2. Time Equation—Two equal expressions for the time.
3. Distance Equation—Two equal expressions for the distance.

The problem should first be analyzed and the facts listed in a table such as given in the illustration below. From the table and the reading of the problem it will be clear whether to form a rate, time, or a distance equation.

Illustration. Two express trains leave New York for Chicago, one two hours after the other. The first train travels 40 miles an hour, the second, 50 miles an hour. In how many hours will the second train overtake the first?

Solution. If we let t equal the number of hours it will take the second train to overtake the first, we can analyze the problem and list the essential facts in a table as here shown:

	distance	rate	time
First Train	40t	40	t
Second Train	50(t-2)	50	t-2

Since the trains travel the same distance, the expressions for the distance are equal, and we have a distance equation, namely,

$$40t = 50(t-2).$$

When the table is used the equation can always be formed from the facts in the column which was filled in last.

This method of treating uniform motion problems has the advantage of:

1. Teaching the major difficulties one at a time.
2. Teaching the pupil how to analyze the problem.
3. Leading the pupil to see that the formula, $d = rt$, is fundamental in solving all uniform motion problems.
4. Revealing the basis on which the equation is built, by noting the column which is last filled in in the table.

Type 2. Mixture Problems. Let us consider problems such as the following:

a. A man wishes to make a 100-pound mixture of candy selling at 60 cents a pound, and candy selling at 50 cents a pound, how many pounds of each must he use to sell the mixture at 55 cents a pound?

b. A druggist in preparing a prescription finds that he needs a chemical 5% acid. He has a one-ounce container filled with a 15% chemical. How much water must he add to make a 5% chemical?

The development leading to the solution of such problems should be as follows:

A. *Providing a Knowledge of Fundamental Facts.* These facts are:

1. *Ability to express in symbols the words and phrases used in mixture problems.* Certain arithmetic abilities are necessary for the solution of such problems. It is important to know if the pupil possesses them. The pupil must also know how to write certain phrases by use of symbols. He must know how to express a certain per cent as a decimal. How to find a certain per cent of a number, how to express a certain per cent of a number x, how to express a certain per cent of a number x increased by n.

2. *Ability to express one variable in terms of the others.* The pupil should next be confronted by such exercises as the following:

a. A mixture of two kinds of candy consists of 25 pounds. If x represents the number of pounds of one kind, how can we represent the number of pounds of the second kind?

b. A man sells n pounds of nuts for 40 cents a pound and 100-n pounds for 30 cents a pound.

How many pounds of both kinds are there?

What is the total cost of each kind?

c. There were x gallons of alcohol 80% pure mixed with 30 gallons of alcohol 65% pure to form a mixture 70% pure.

What was the amount of pure alcohol in the 80% solution?
What was the amount of pure alcohol in the 65% solution?
What was the amount of pure alcohol in the mixture?

B. *How to Analyze the Problem.* The pupil should next be given several problems where the questions are as detailed as in the following illustrations. It is typical of analyses of all such problems.

Illustration 1. Mr. A. has tea worth 45 cents a pound and tea worth 60 cents a pound. How many pounds of each kind must be taken to make a mixture of 50 pounds worth 55 cents a pound?

a. How can we represent the number of pounds of 45-cent tea to be taken?

b. How, in terms of (a), can we represent the number of pounds of 60-cent tea to be taken?

c. What is the total cost of the 45-cent tea? the 60-cent tea?

d. Write an expression for the total cost of the two kinds of tea in terms of the answers to (c).

e. What is the total cost of the mixture?

f. Write an equation that the above facts suggest.

Illustration 2. Iron ore containing 10% pure iron is mixed with iron ore containing 6% pure iron to make a mixture of 100 tons containing 9% pure iron. How many tons of each kind of ore is used in making the mixture?

a. How can we represent the number of tons of the 10% ore to be taken?

b. How can we represent, in terms of (a), the number of tons of the 6% ore to be taken?

c. How much pure iron is there in the 10% ore? the 6% ore?

d. Write an expression for the total amount of iron in the two kinds of ore, using the answers to (c).

e. How much pure iron is there in the mixture?

f. Write an equation that the above facts suggest.

C. *Organizing the Facts to Solve the Problem.* The solution of mixture problems depends upon one of the following formulas:

1. $C_1 + C_2 = C_m$, where C_1 is the total cost of one commodity, C_2 the total cost of a second commodity, and C_m the total cost of the mixture.

2. $Q_1 + Q_2 = Q_m$, where Q_1 is the amount of pure substance in one material, Q_2 the amount of pure substance in the other material, and Q_m the amount of pure substance in the mixture.

The problem should first be analyzed and the facts listed in a table such as is given below. From the table it will be a simple matter to form an equation by using one of the preceding formulas.

Illustration. How many tons of iron ore 6% pure must be mixed with iron ore 12% pure to obtain a mixture of 60 tons that is 10% pure?

Solution. If we let x equal the number of tons of 6% ore required, and 60-x the number of tons of 12% ore required, we can analyze the problem and list the facts in the following table:

	Amount of ore	Per Cent of Pure Iron	Amount of Pure Iron	
First Ore	x	6	.06x	Q_1
Second Ore	60-x	12	.12 (60-x)	Q_2
Mixture	60	10	.10 (60)	Q_m

Since the amount of pure iron in the one ore plus the amount of pure iron in the second ore is equal to the amount of pure iron in the mixture, we can use the formula $Q_1 + Q_2 = Q_m$, and substitute in it to have the equation,

$$.06x + .12(60\text{-}x) = .10(60).$$

Type 3. Profit and Loss Problems. Let us consider problems such as the following:

1. A dealer sold a suit of clothes for $46.50 thereby making a profit of 30% of the cost. How much did the suit cost him?
2. A real estate dealer learns that a customer is willing to pay $11,500 for a house. At what price must the agent buy the house to make a profit of 15% of the cost?
3. A dealer wishes to make a profit of 25% of the selling price on an article. What price must he pay for the article if he wishes to sell it for $2.50?

The development leading up to the solution of such problems should be:

A. *Providing a Knowledge of Fundamental Facts.* These facts are:

1. *Ability to express in symbols the words and phrases of profit and loss problems.* The pupil should have the ability to express by means of symbols such words and phrases as,

Profit	Cost plus gain
Cost	Selling price minus gain
Loss	Selling price minus cost
Gain	A certain per cent of the cost
Selling Price	A certain per cent of the selling price

2. *Ability to express one variable in terms of the others.* The pupil should next be confronted by such exercises as:

a. What would the profit be if:
The cost were $60 and the profit 15% of the cost?

The selling price were $125 and the profit 20% of the selling price?
The profit were 10% of the cost, *c*?
The profit were 25% of the selling price, *s*?

b. What would be the cost if:
The profit were $60, or 20% of the cost?
The loss were $120, or 30% of the cost?
The gain were $25, or 10% of the cost?
The profit were $240, or 120% of the cost?

c. What would be the selling price if:
The profit were $50, or 20% of the selling price?
The profit were $40, or 125% of the selling price?
The loss were $12 or 60% of the selling price?

B. *How to Analyze the Problem.* Next should be considered several problems where the detailed questions will aid in the analysis.

Illustration. A man sold his radio set for $140, thus losing 20% of the cost. Find what the radio set cost him.

a. How can we represent the cost?
b. How can we represent the loss in terms of the cost?
c. Since the selling price is equal to the cost minus the loss, write an expression for the selling price using the answers to (a) and (b).
d. Since the selling price is $140, write an equation from the above facts.

C. *Organizing the Facts to Solve the Problem.* The solution of problems about profit and loss can be solved by aid of one of the following formulas:

1. $C + G = S$, where C is the cost, G the gain or profit, and S the selling price.
2. $C - L = S$, where C is the cost, L the loss, and S the selling price.

The problem should be analyzed and the facts listed in a table such as is given below. From the table the values can be substituted in one of the above formulas.

Illustration. A merchant sold an article for $2.60 thus making a profit of 20% of the selling price. What was the cost of the article?

Solution. If we let C equal the cost of the article, we can analyze the problem and list the facts as follows:

Cost	Profit	Selling Price
C	.20 of $2.60	$2.60

These terms suggest the formula $C + G = S$. Substituting in this formula, we have the equation

$$C + .20 \times \$2.60 = \$2.60.$$

Type 4. Geometry Problems. Let us consider problems of the following nature:

1. The width of a rectangle is equal to two-thirds of its length. If the perimeter is 84 feet, find the length and the width of the rectangle.
2. Find the size of each angle of a triangle if two of the angles are equal and the third angle is five times the sum of the other two angles.
3. Find the base and the altitude of a triangle containing 96 square inches if the height is 4/3 of the base.

The development leading to the solution of such problems should be:

A. *Providing a Knowledge of Fundamental Facts.* These facts are:

1. *Ability to express in symbols the words and phrases used in geometry problems.* The pupil should be able to express in symbols such words and phrases as,

Perimeter	Height
Area	Circumference
Length	Base times the height
Width	Pi times the diameter

2. *Ability to express one variable in terms of the others.* The pupil should next be confronted with such exercises as these.

a. The width of a rectangle is two-thirds of its length.
How can we represent the length?
How can we represent the width in terms of the length?

b. The base of a triangle is four feet more than twice the height.
How can we represent the height?
How can we represent the base in terms of the height?

c. The circumference of a circle is equal to pi times the diameter.
How can we represent the diameter?
How can we represent the circumference in terms of the diameter?

B. *How to Analyze the Problem.* Several exercises such as the following should then be considered with the questions as detailed as here stated.

Illustration. The length of a tennis court is six feet more than twice the width. Find the dimensions if the perimeter of the court is 228 feet.

a. How can we represent the width of the court?
b. How can we represent the length of the court in terms of the width?

c. Write an expression for the perimeter using (a) and (b).
d. Since the perimeter is 228 feet, write an equation from the above facts.

C. *Organizing the Facts to Solve the Problem.* The solution of geometry problems depends upon such formulas as the following:

1. $A = lw$	5. $\angle A + \angle B + \angle C = 180^\circ$
2. $P = 2l + 2w$	6. $A = \pi r^2$
3. $A = \frac{1}{2}bh$	7. $A = s^2$
4. $C = \pi d$	8. $V = lwh$

The problem should first be analyzed and the facts listed in tabular form. Certain words will suggest the formula required. The problem is then solved by substitution in the formula.

Illustration. The length of a rectangle is 6 feet more than twice the width. If the area contains 360 square feet, find the length and the width of the rectangle.

Solution. If we let w equal the width of the rectangle, we can prepare a table as shown here:

Area	Length	Width
360	$2w + 6$	w

Substituting these values in the formula $A = lw$, we have,

$$360 = w\ (2w + 6).$$

Other types can be treated in a similar fashion. The advantages of treating the solution of problems as briefly described here, are:

1. The fundamental facts necessary to an understanding of a particular type of problem are systematically taught.

2. The pupil is taught to analyze the problem.

3. A method of organizing facts is developed in the pupil.

4. The pupil is lead to see that a formula or a group of formulas is fundamental to the solution of any given type of problem.

This method eliminates the catch-as-catch-can method employed by many text-books and many teachers. A pupil having been instructed in this method of solving problems has learned a method of great service to him when solving a miscellaneous group of problems. At once the problem is classified as to type and the facts about such problems recalled. The writer has found the method most effective over a period of years.

Another Phase of the Geometry Situation

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In this paper it is assumed that:

1. A course of study must be elastic enough to be adaptable in its content and in its teaching to the abilities of the pupils who are to take it.
2. The general character of content in the ninth grade will be algebraic and in the tenth grade geometric and demonstrative.
3. The only way to make a content effective is to develop it progressively and use it over a considerable period of time.
4. A specific content must be of the greatest possible value to the pupil whether or not this value lies in subject matter or in its method of learning or in both.

These assumptions clearly indicate that acceptable courses of study will not be found by fiat but as the result of experimentation and may differ where the conditions are different.

The normal development of tenth grade courses has been retarded by the Euclidean tradition and the effect of extra mural examinations. At present interest in geometrical curricula has been revived because of the proposal to define authoritatively a syllabus of a one year course in both plane and solid demonstrative geometry as a substitute for the present course in plane geometry. In some quarters the proposal is construed as a curriculum resembling the present minor requirement in plane and solid geometry of the College Entrance Examination Board.

The aim of such a proposal is obviously to add the values of extended specific information and space intuition to present values. These values are considered great enough to justify the attempt to secure them from all the pupils now taking plane geometry rather than the relatively few who take solid geometry.

It is certain that the proposal must include a very considerable decrease in the content of present courses; the decrease in time alone being one-third.

Assuming the situation to be as stated, and in the absence of more

specific information about it, the proposal is open to the following objections:

1. It is not likely that the syllabus of the new course will be the outcome of actual classroom experimentation and therefore

2. The new course will not overcome the difficulties that exist now but will possibly increase them owing to the increase in objectives.

Some of the difficulties in the present ninth-tenth grade situation are:

1. The language and expression difficulty. This is due both (a) to the wealth of new ideas requiring specific technical words precisely defined and (b) to the cumbrous traditional method of stating relations. The first of these has been attacked and is slowly yielding to pressure. Better teaching methods will decrease the difficulty still further. It cannot, however, be reduced beyond a certain necessary minimum. The second part of this general difficulty varies widely from school to school at the present time. It is due to the fact that the Greeks did not possess operative symbols for numbers or number relations. In a sense pupils are required to live mathematically in the time of the Greeks and ignore the intervening human progress and the skills that they already know from their preceding year's work. The relations to be expressed in geometry are mostly numerical (largely measurement numbers) and a system of symbols that does not artificially and needlessly introduce difficulties must be used.

2. The adjustment difficulty between the ninth and tenth grade courses. The present distinctive nature of tenth grade demonstrative geometry creates most of this difficulty although ninth grade content can be improved in the same direction. The unsatisfactory outcome of attempts at "fused" courses, where the traditional character of geometry was preserved, shows both the pressure to meet this difficulty and the nature of its solution. Even with the best articulation possible there will always be enough of this difficulty to cause concern.*

3. The loss or lack of use difficulty. The algebra-geometry-alge-

* The 6-3-3 division of the grades intensifies it. The growing custom of continuing algebra in the tenth grade and undertaking geometry in the eleventh is an attempt to escape particularly from this situation while minimizing other difficulties through pupil elimination and the increased maturity of those remaining. This is a justifiable plan where the elimination is small.

bra sequence with the present minimum use of algebra in geometry produces large and unnecessary losses. These losses must be minimized. It cannot be done unless the greatest possible legitimate use is made in grade 10 of the skills developed in grade 9, and so on.

4. The unevenness of logical standards difficulty. This one is inherited and traditional. Current practice is uneven as between (a) explicit and tacit assumptions, (b) proof and assumption, (c) proof and proof, (d) text and text, (e) school and school, and (f) teacher and teacher in the same school. Of these possibly the greatest difficulty for the pupil lies in the precision required in one proof and the looseness tolerated in another. Real vigor is of course out of the question. That is no reason, however, why the necessary assumptions should not be made explicitly and the work that is attempted be correctly derived from them. It might appear that a satisfactory solution of this difficulty could be obtained from the authoritative standards of a committee. That would require the production of a practical text—a form of report obviously objectionable. As to what must be assumed and what demonstrated experimentation alone can furnish the answer.

The results that appear to me to follow from the preceding analysis are:

1. Changes in the present course in tenth grade geometry are essential.

2. The proposal for a year of plane and solid geometry be deferred until the difficulties of the present situation are minimized through an improved course of study that incorporates the results of actual classroom trial, unless the proposed course can also be tried out at the same time.

I am of the opinion that a trial of content according to the following description would be worth while:

Ninth Year Course

Elementary Algebra

I The introduction of, or greater emphasis upon:
- A. Relations of Variables
    1. Comparisons
    2. Discovery and Formulation
    3. Ratio, Proportion, and Variation (New Treatment)

4. Linear Relations
5. Parabolic Relations (One Linear, one Quadratic)

B. Methods of Calculation
1. Exponents and Radicals
2. Logarithms (Scales and Tables) (Slide Rules)
3. Graphical Methods

C. Trigonometry

D. Quadratic Equations

II Decreased emphasis upon, or the elimination of:

A. Certain Types of Verbal Problems and the Total Quantity of Such Material.

B. Considerable Factoring

C. Unnecessary Complications Arising from Fractional Forms

Tenth Grade Course

Demonstrative Geometry

I. General introduction of algebraic symbols into demonstrative geometry for use in *discovery, demonstration,* and *formulation* of results.

II. Larger emphasis upon *types of discovery and demonstration* and lessened emphasis upon the demonstration of *single theorems.* A reduction in the total number of theorems. A new treatment of constants, variables, and limits in geometry leading to a discussion of one or more theorems. Greater emphasis upon loci.

III. The space extension of some of the assumptions, definitions, theorems, and exercises. The problem of representation of a solid figure. Sections of a solid.

IV. The use of trigonometry in the extension and formulation of theorems. Its use in calculation.

V. Calculation by logarithms and tables.

With respect to the ninth grade course I am confident that no particular difficulties will be found.* The tenth grade course is the stumbling block at the present time.

* There is even a satisfactory text now in print that embodies substantially such a course.

A Study of Individual Differences as They Effect Mathematics Classes

By MABEL SYKES
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THE writer is making a study of individual differences as they effect mathematics teaching in high schools and invites correspondence on the subject. Among the points to be covered are:

1. In elementary (ninth grade) algebra and in plane geometry are weaker and stronger pupils taught in the same classes or is some special provision made for weaker pupils?

2. If there is a difference in the composition of classes, on what basis is the selection made?

3. What is the composition of the weaker class? Is it made up mostly of young people of the same age but with lower IQ's or a different inheritance than those in the stronger class, or is it made up largely of older young people who for one reason or another are behind their fellows and who may have acquired other interests than those connected with school work?

4. How does the work given to the weaker class differ from that given to the stronger class? Is it mostly a matter of covering less ground or is the plan of the work different? In the latter case, how?

5. If pupils are not classified as suggested above, is any special provision made for their needs in the regular classes, and if so, what?

6. What is done with pupils of the types mentioned under 3 above when they elect the more advanced mathematics given in the high school, the work intended primarily for those who hope to go to the universities or technical schools?

7. Are any other methods used to care for such pupils?

An increasingly large number of young people of the types mentioned above are to be found in many if not in most high schools. They do not appear to profit by the work as previously given. The problem of what is to be done for them is becoming more and more pressing. Hence this survey. The writer hopes to report findings later. Address correspondence to 5546 Blackstone Avenue, Chicago, Illinois, Hyde Park Station.

Debates Versus Examinations in the History of Mathematics

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THERE ARE MANY things that make the older methods of examinations in the History of Mathematics unsatisfactory and of doubtful portent for the future. My first examination in this subject was under Dr. Myers of the University of Chicago. I must admit that as we galloped through the centuries studying the lives and works of many Greeks, Arabs, Hindus, Egyptians, Babylonians, Italians, French, English, German, Chinese, and Japanese that I was hopelessly confused and much of the pleasure of the course was smothered by the thoughts of a final examination in this subject. Dr. Myers, being a really human teacher, knew our difficulties and gave us an "Open-book" examination; however, the time was limited and the questions were long and numerous. Had we not possessed a fair knowledge of the subject, we could not have made a passing grade with the help of the open book. In my graduate work with Dr. David Eugene Smith at Columbia University, the same barrier of a final examination was before me. We had two hours for the examination and we had two questions; one for each hour. The first question was: "Name the ten leading Mathematicians of all time. List them according to the value of their contribution to Mathematics and give the reasons for your choice and your order of arrangement." Now, don't you think this is a good question? I do. But I wonder how Dr. Smith graded the various choices and orders of arrangement. The question was surely a thought provoking, searching one.

My own experiences in these two classes have led me to try to devise some other scheme of examining my students in the History of Mathematics. For six years I have used debates as substitutes for examination. At the beginning of the term I explain my method of examination to the class and all through the course we select the debatable subjects and list them. At mid-term and at the close of the term, each member of the class submits three debatable subjects bearing on the history we have studied. All the subjects

presented are placed before the class and a subject for the debate is chosen by the class. A class period is one hour, and since my class has never exceeded six in number, each member has something like ten minutes for his debate. We divide the time in such a way that there will be a few minutes for rebuttal and a rejoinder for the affirmative. These debates are conducted according to laws of debating teams and the decision is in the hands of a critic judge. I am always present and take full notes on the preparation, choice of material, general knowledge of the subject, as shown by each debater. The grades in these debates are given about the same value in the final grade that an examination grade is given.

SOME OF THE SUBJECTS PRESENTED BY CLASS OF 1930

1. Resolved, that the mathematical contributions of Archimedes have been more powerful in the world than the mathematical contributions of Euclid.
2. Resolved, that the mathematics of the Greeks has been more beneficial to mankind than the mathematics of the Hindus.
3. Resolved, that English contributions to mathematics in sixteenth and seventeenth centuries surpass the contributions of the Greeks and Hindus together.
4. Resolved, that the Alexandrian School left a deeper imprint for mathematics than did the school at Bagdad.

Many of the subjects submitted are not debatable; for example, debate number one read: "Resolve that Archimedes gave more mathematics to the world than did Euclid." There is just one side to that question. After the debate is over, I have asked the students to list the advantages and disadvantages of this type of examination.

ADVANTAGES

1. Each debater must defend his own statements by references.
   a. Careful selection of materials is made
   b. Many more references are read and digested
2. Comparisons of men and periods are emphasized. A more careful study of each character is made.
3. The period is studied from the general view-point and then in detail.
4. In selection of subjects, essentials are separated from non-essentials.
5. Much more enthusiasm is created.
6. Major events and contributions are intensified.
7. Each student must know both sides of the question in order to be able to answer his opponent.
8. A valuable training comes through standing before the class and giving the argument.
9. Organizing material for a debate in the History of Mathematics is a more powerful type of review than preparaing for an examination.

10. In preparing for an examination, the text alone is largely considered; in preparing a debate, all available material is consulted.

Disadvantages

1. Each member has too little time to express himself.
2. A debate limits the review to two phases and necessitates the greatest care in the choice of subject.
3. It takes more time to prepare for the debate than it does to prepare for an examination.
4. All major events and contributions cannot be used in one debate.

These disadvantages can be overcome by having a debate at the close of each period, and by dividing the final debate in as many parts as the size of the class will justify. In each final debate the subject should bring into play the whole field of the History of Mathematics.

Johann Kepler

1571-1630

In 1594 when Kepler left the University of Tübingen, he reluctantly accepted an appointment to lecture on mathematics and astronomy in the Gymnasium at Gratz, feeling that his preparation in these subjects was inadequate for the position. His doubts were borne out by the evidence that in the first year of his work he had few students in mathematics while the second year he had none. At Gratz, Kepler turned his attention to astrology which was then considered essential to an astronomer, and he became uncannily successful in his predictions. It was at this time that he formulated a theory regarding a connection between the planets and the regular solids thus convincing himself that since there were but five of the solids there could be but six planets.

In 1598, religious difficulties led to the withdrawal of students from Gratz and Kepler and the other teachers were banished. Kepler was then invited to become assistant to the astronomer Tycho Brahe who had recently left Denmark to be court astronomer to the Emperor Rudolph II in Prague. Here one of Kepler's duties was the compilation of a yearly almanac. The sale of the almanac and the horoscopes which he cast supplemented a salary which was frequently in arrears and provided funds for other publications. Whether Kepler himself believed in his predictions is doubtful. It is known however that he sometimes prefaced his horoscopes with the statement that what followed might or might not be true.

Tycho Brahe had made many observations of the positions of the planets, and on his death in 1601, Kepler was given the task of working over this data and of completing the Rudolphine Tables. In the course of this work, Kepler formulated his laws of planetary motion:

I. Every planet moves in an ellipse, the sun being at one focus.
II. A line joining the center of the sun to the center of a planet sweeps out equal areas in equal intervals of time.
III. The square of the time of one complete revolution of a planet about the sun is proportional to the cube of its mean distance from the sun.

The first two were published in 1609. The third, which is only approximately correct, was announced in 1619.

Kepler's work in mathematics included an assumption of the law of continuity in geometry and an extensive use of the method of exhaustions in which a circle was considered as being composed of infinitely many triangles with a common vertex at the center of the circle, and a sphere of infinitely many pyramids. A dispute over the volume of a wine cask led him into an investigation of the subject of gauging and the publication of a work on stereometry which anticipated the invention of the integral calculus. Kepler was one of the first to utilize logarithms and he dedicated a series of ephemerides published in 1620 to Napier apparently not knowing of Napier's death in 1617.

NEW BOOKS

Mathematik und Bildende Kunst. By Walther Lietzmann, 149 pp. Ferdinand Hirst in Breslau, 1931. M.6.80.

"Scientia est unum et ars est aliud"—science is one thing, and art is another—said the Latins, and this is the general conception so common to most of the laymen. Almost every day we hear that art and mathematics do not mix. Recently we overheard this remark: "Try to mix oil and water—you cannot and will not, and 'your' mathematics has no place in art."

We shall not dwell on the questions "what is art?" and "what is mathematics?" What difference can it make what they are?—We may go on for centuries, and we shall never arrive at a satisfactory definition for all concerned, they are not definable. Our reply to the uninformed layman will be in Latin also: "ars sine scientia nihil est"—art without science is nothingness,—emptiness,—an interstellar vacuum.

What is it that makes a painting attractive, what elements have to be stressed in order that a building may appeal to us in its design, and what principles make a statue perfect from the esthetic point of view? Are these questions emotional in their nature? The layman and perhaps the average artist will give an affirmative reply. But let us disappoint them,—or better let Lietzmann do so.

In the limited space of his new book Lietzmann shows the evolution of architecture from the most primitive forms. We see how the simplest architectural design gradually developed from triangles, how with the advent of time more complicated forms, such as the circle, cylinder, ellipse, and conics of revolution have become frequent. We are amazed to learn that almost every period in architecture

can be identified and is recognized by its characteristic geometric figure and its component elements. Even if we disregard the mechanical principles of construction and the elementary requirements underlying the science of architecture, the forms of building design have an evolution of their own, and when taken as a whole the architecture of the entire world is an open book of geometry. Moreover, what surprises one most is the fact that there is uniformity in this evolution. It is almost as uniform as a natural law. The angles of the isosceles triangles that adorn the façades of the temples of Acropolis are not accidental in size, nor are the shapes of the windows of Notre Dame, of the Cathedral of Cologne, or of the Chapel of Christ College at Oxford.

If we study the patterns of the designs of pottery, textiles, tapestry, wood carvings, armor plates, diamonds,—from the Stone Age up to the present day we again have a science of its own, a mathematical science—the theory of groups. If we study paintings—from the time of the Cro-Magnon man on, we can be convinced that the principles of symmetry, of perspective of all types are the underlying elements that make those "things of beauty—a joy forever." Should we go on and enumerate the principles of proportion, ratio, the "golden section"? The layman will say that no artist knows or is aware of all these. Dr. Lietzmann promptly replies that Dürer and Rembrandt, as well as Leonardo da Vinci and many others knew, used, and have written about them. Art becomes art when mathematics enters into it. Music without mathematics is a cacaphony. Painting without mathematics is lifeless,—simply a conglomeration of colors without order, symmetry, or idea.

The book is luxuriously illustrated. Examples are drawn from masterpieces all over the world. What can we say, when we, here in America, have not paid any attention to our own art, when few mathematics teachers are aware of the fact that we have here, in New York, in the Metropolitan Museum of Art and in the American Museum of Natural History the best and the most costly collections that have been mentioned and discussed by Lietzmann in this book. Could not a pupil be stimulated, to a more intense study of geometry, as well as mathematics in general, if instead of spending hours on solving meaningless (to him) originals, he could wander under guidance in the spacious halls of a museum, or be shown buildings in his own town, or shown reproductions of paintings, sculptures, and other forms of art, in the classroom—with the accompaniment of an intelligent and interesting discussion. Almost every pupil owns or can own a small photographic camera. Have we ever realized the fact that even photography is subject to the same principles as painting?

We can only welcome a book of this type. It is only regrettable that there are no books of this kind in English. A teacher of mathematics who has a command of German should by all means get this book and read it. It will stimulate him to devote his geometry class period to a survey of the histories of art and culture—and these are the main parts of the history of Mankind. Surely this will give the pupil a different view of mathematics.

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